JHEP HYPER VERSION

RECEIVED: *September 29, 1998*, ACCEPTED: *October 10, 1998*

On asymptotic Hamiltonian for $SU(N)$ matrix theory

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ABSTRACT: We compute the leading contribution to the effective Hamiltonian of $SU(N)$ matrix theory in the limit of large separation. We work with a gauge fixed Hamiltonian and use generalized Born-Oppenheimer approximation, extending the recent work of Halpern and Schwartz for $SU(2)$ [8]. The answer turns out to be a free Hamiltonian for the coordinates along the flat directions of the potential. Applications to finding ground state candidates and calculation of the correction (surface) term to Witten index are discussed.

KEYWORDS: M(atrix) Theories, M-Theory.

JHEP10(1998)018

Contents

1. Introduction

We study the maximal supersymmetric gauge quantum mechanics. This model was first considered in papers [1, 2, 3]. Later this Hamiltonian found applications in the physics of supermembranes [4, 5] and D-particles [6]. There is a remarkable conjecture, called Matrix theory, that the same Hamiltonian gives a nonperturbative description of M-theory [7]. This conjecture implies the existence of a unique bound state at threshold for each $SU(N)$ gauge group where $N \geq 2$ (though the existence of the state was conjectured earlier).

The $SU(2)$ case was recently studied in a number of papers [8, 9, 10, 11]. In [9] it was proved, using the computation of Witten index, that at least one ground state exists in $SU(2)$ model. Some progress also have been achieved in the general $SU(N)$ case [12, 13, 14]. In [13] the principal contribution to Witten index was computed. Calculation of the full index still remains an open question. In [8] an asymptotic form of a wave function was proposed as a candidate ground state.

The present paper extends the ideas of [8] and [11] to higher N's. We consider the model in the asymptotic region where the coordinates along the flat directions of the potential become large. Our method is generalized Born-Oppenheimer approximation introduced in [8]. The authors of [8] tackle the problem of gauge invariance by working with a complete set of gauge invariant variables. It would be rather hard to implement this idea for higher N's. Instead, we apply the gauge fixing procedure of [5]. The result of our computation is that in the leading order the dynamics of flat coordinates is described by the free Hamiltonian. Possible potential term vanishes due to cancellations between many different terms. This result is not surprising, it is generally expected in supersymmetric theories. It is in agreement with $SU(2)$ calculations of [8, 11, 9]. In the last section of the paper we discuss possible applications of the main result.

2. The model and preliminaries

In this section we introduce the model and do some preliminary technical work needed for the Born-Oppenheimer approximation. Namely, we write down the (partially) gauge fixed Hamiltonian, split it into 4 basic parts: free Hamiltonian for slow (Cartan) degrees of freedom, bosonic and fermionic oscillators, and the interaction part. After that we explain the quantization of the oscillators part. We refer the reader to the seminal paper [5] for all the details of the gauge fixing procedure.

The Hamiltonian of matrix theory is that of the 10D super Yang-Mills theory dimensionally reduced to 0+1 dimensions (in $A_0 = 0$ gauge). One can write it in the following form:

$$H = \frac{1}{2}\pi^\mu_A\pi^\mu_A + \frac{1}{4}f_{ABC}\phi^\mu_B\phi^\nu_C f_{ADE}\phi^\mu_D\phi^\nu_E - \frac{i}{2}f_{ABC}\phi^\mu_A\Lambda_{B\alpha}\Gamma^\mu_{\alpha\beta}\Lambda_{C\beta}\,, \tag{2.1}$$

where ϕ^μ_A, π^μ_A are real bosonic variables and $\Lambda_{A\alpha}$ are real fermionic variables. The lower capital Latin indices are that of the adjoint representation of a real compact Lie algebra $\mathbf{g}$ with totally antisymmetric structure constants f_{ABC}. Denote by G the Lie group of $\mathbf{g}$. The indices $\mu,\nu = 1,\ldots,9$, $\alpha = 1,\ldots,16$ correspond to vector and real spinor representations of the group $spin(9)$ respectively. The $SO(9)$ gamma matrices Γ^μ are assumed to be real and symmetric. The canonical commutation relations between the variables are

$$[\pi^\mu_A, \phi^\nu_B] = -i\delta^{\mu\nu}\delta_{AB}\,, \qquad \{\Lambda_{A\alpha}, \Lambda_{B\beta}\} = \delta_{\alpha\beta}\delta_{AB}\,.$$

From now on we set $G = SU(N)$ that corresponds to the Matrix theory with the "center of mass" degrees of freedom being excluded. The Lie algebra $\mathbf{g} = su(N)$ consists of the traceless antihermitean $N\times N$ matrices ϕ. An invariant, positive definite inner product on $\mathbf{g}$ is defined as $(\phi_1,\phi_2) = -2Tr(\phi_1\phi_2)$. The Cartan subalgebra consists then of diagonal matrices $\phi\in\mathbf{g}$. Let matrices T_A, $A = 1,\ldots,N^2-1$ be an orthonormal basis in $su(N)$ with respect to this inner product such that the indices $A = 1,\ldots,N-1$ correspond to the Cartan subalgebra. Then $f_{ABC} = (T_A,[T_B,T_C])$.

Now we will briefly explain the gauge fixing procedure. Given any element $\phi\in g$ there exist a unique matrix D such that $\phi = UDU^{-1}$ for some $U\in G$ and D is of the form

$$D = i\begin{pmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \lambda_N \end{pmatrix}, \tag{2.2}$$

where λ_n are real numbers such that

$$\sum_{n=1}^{N} \lambda_n = 0\,, \qquad \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_N\,. \tag{2.3}$$

The transformation U is defined up to multiplications by an arbitrary element of the Cartan subgroup. Thus, it is clear that we can perform a partial gauge fixing by requiring that ϕ^9 lies within the Cartan subalgebra and has the form (2.2) of the matrix D above. This particular form corresponds to a fixed Weyl chamber within the Cartan subalgebra. The residual gauge group is then $U(1)^{N-1}$. This gauge fixing procedure is described in details in [5]. Following [5] we adopt the convention that the indices $i, j, k, \ldots$ are summed from 1 to $N-1$ (the indices of Cartan subalgebra) , while capital indices $I, J, K, \ldots$ from the middle of the alphabet run from N to N^2-1 (the indices of the subspace spanned by the roots). Also we assume that the indices $a, b, c, \ldots$ run from 1 to 8 and correspond to the first eight coordinates in R^9. Finally, let us adopt a new notation for the bosonic Cartan variables $D_i^\mu \equiv \phi_i^\mu$ (that will be convenient in the future when we split the variables into the "fast" and "slow" ones). The gauge fixed Hamiltonian is $H = H_0 + H_B + H_F + H_4$, where

$$H_0 = -\frac{1}{2}\frac{\partial^2}{\partial D_i^\mu \partial D_i^\mu}\,, \tag{2.4}$$

$$H_B = -\frac{1}{2}\frac{\partial^2}{\partial \phi_I^a \partial \phi_I^a} + \frac{1}{2}\Omega_{IJ}^2 \phi_I^a \phi_J^a\,, \tag{2.5}$$

$$H_F = -\frac{i}{2}\Lambda_I z_{IJ}^\mu \Gamma^\mu \Lambda_J\,, \tag{2.6}$$

$$\begin{aligned} H_4 = {} & \frac{1}{4} f_{AIJ} f_{AKL} \phi_I^a \phi_J^b \phi_K^a \phi_L^b + \frac{1}{2} f_{AiJ} f_{AKL} (D_i^a \phi_J^b - D_i^b \phi_J^a) \phi_K^a \phi_L^b - \\ & - \frac{1}{2} f_{AiJ} f_{AkL} D_i^a D_k^b \phi_J^b \phi_L^a - \frac{i}{2} f_{IAB} \phi_I^a \Lambda_A \Gamma^a \Lambda_B + \frac{1}{2}(w^t w)_{IJ} \hat{L}_I \hat{L}_J\,. \end{aligned} \tag{2.7}$$

The following notations are used above:

$$\Omega_{IJ}^2 = (z_\mu^t z_\mu)_{IJ}\,, \qquad z_{IJ}^\mu = D_i^\mu f_{iIJ}\,, \qquad w = (z^9)^{-1}\,,$$

$$\hat{L}_I = -f_{IBC}\left(i\phi_B^a \frac{\partial}{\partial \phi_C^a} + \frac{i}{2}\Lambda_{B\alpha}\Lambda_{C\alpha}\right).$$

This Hamiltonian is self-adjoint with respect to the measure $\prod_{a,A} d\phi_A^a \prod_i dD_i^9$ (the Faddeev-Popov Determinant was eliminated by redefining the Hilbert space).

The terms H_B and H_F are Hamiltonians of bosonic and fermionic harmonic oscillators. To diagonalize these Hamiltonians, note that the eigenvectors of the matrices z_{IJ}^μ are the complex root vectors E_{mn}^I $(m, n = 1, \ldots, N\,, m \neq n)$. These eigenvectors satisfy

$$z_{IJ}^\mu E_{mn}^J = i(\lambda_m^\mu - \lambda_n^\mu) E_{mn}^I\,, \tag{2.8}$$

$$(E_{mn}^I)^* = E_{nm}^I. \tag{2.9}$$

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Here λ^μ_m are eigenvalues of matrices $D^\mu = \sum_i D^\mu_i T^i$. The vectors E_{mn} define an orthonormal basis of a root subspace:

$$\sum_I (E^I_{mn})^* E^I_{pq} = \delta_{mp}\delta_{nq} \qquad (m \neq n, p \neq q)\,, \tag{2.10}$$

$$\sum_{m\neq n} (E^I_{mn})^* E^J_{mn} = \delta_{IJ}\,. \tag{2.11}$$

Note the equality $E_{mn} \equiv E^I_{mn} T^I = \frac{i}{\sqrt{2}} e_{mn}$, where e_{mn} is $N \times N$ matrix with 1 on the m,n-th place and zeros elsewhere. These matrices satisfy the following commutation relations:

$$[E_{mn}, E_{pq}] = \frac{i}{\sqrt{2}}(\delta_{np}E_{mq} - \delta_{mq}E_{pn})\,. \tag{2.12}$$

The eigenvalues of H_B and H_F (as we will see further) are

$$r_{mn} = \sqrt{\sum_{\mu=1}^{9} (\lambda^\mu_m - \lambda^\mu_n)^2}\,. \tag{2.13}$$

Thus, we can introduce new variables ϕ^a_{mn}, Λ^{mn}_α, $m, n = 1, \ldots, N$, $m \neq n$ so that

$$\phi^a_I = \sum_{m\neq n} \phi^a_{mn} E^I_{mn}\,, \qquad \Lambda_{I\alpha} = \sum_{m\neq n} \Lambda^{mn}_\alpha E^I_{mn}\,. \tag{2.14}$$

The commutation relations for fermions are now

$$\{\Lambda^{mn}_\alpha, \Lambda^{pq}_\beta\} = \delta_{\alpha\beta}\delta^{np}\delta^{mq}$$

and it is natural to set $(\Lambda^{mn})^\dagger = \Lambda^{nm}\; m < n$. However, to diagonalize H_F some more work needs to be done. Namely, one should use the *spin*(9)-rotated fermions

$$\begin{aligned} \tilde\Lambda^{mn}_\alpha &= R^{mn}_{\alpha\beta}\Lambda^{mn}_\beta\,, \qquad m < n\,,\\ \tilde\Lambda^{mn}_\alpha &= R^{nm}_{\alpha\beta}\Lambda^{mn}_\beta\,, \qquad m > n\,, \end{aligned}$$

where

$$R^{mn}_{\alpha\beta} = \frac{r_{mn} + (\lambda^\mu_m - \lambda^\mu_n)\Gamma^9\Gamma^\mu}{2r_{mn}(r_{mn} + \lambda_m - \lambda_n)}\,, \qquad m < n$$

is an orthogonal matrix. Here and everywhere λ_n with the suppressed upper index stands for λ^9_n. In terms of $\tilde\Lambda^{mn}$ variables H_F can be written as

$$H_F = \sum_{m<n} r_{mn}\left((\tilde\Lambda^{mn}_+)^\dagger\tilde\Lambda^{mn}_+ + \tilde\Lambda^{mn}_-(\tilde\Lambda^{mn}_-)^\dagger - 8\right)\,, \tag{2.15}$$

where $\Lambda_\pm$ denotes the chiral components taken with respect to Γ^9.

Using (2.8)–(2.12) we can rewrite all parts of the Hamiltonian in terms of the variables D^μ_i, $\Lambda_{i\alpha}$, ϕ^a_{mn}, Λ^{mn}_α. The corresponding expressions for H_0 and H_F are given in (2.4) and (2.15). The Hamiltonian for bosonic oscillators reads now as

$$H_B = -\sum_{m<n} \frac{\partial^2}{\partial\phi^a_{mn}\partial(\phi^a_{mn})^*} + \sum_{m<n} r^2_{mn}\phi^a_{mn}(\phi^a_{mn})^*\,. \tag{2.16}$$

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As one can easily see from (2.15) and (2.16) the ground state energies for the bosonic and fermionic oscillators precisely cancel each other (we have 8 bosonic modes for each $r_{mn}, m < n$). The expression for H_4 in terms of the new variables is rather long. We relegate it to the appendix A in order not to complicate the main text.

The normalized state vector of the oscillators ground state has the following form:

$$
\begin{aligned}
|0\rangle &= |0_B\rangle|0_F\rangle\,,\\
|0_B\rangle &= \left(\prod_{m<n} r_{mn}^4 2^4 \pi^{-4}\right) exp\left(-\sum_{m<n} r_{mn}\phi^a_{mn}\phi^a_{nm}\right),\\
|0_F\rangle &= \prod_{m<n}\prod_{\alpha=1}^{8}\left(\tilde{\Lambda}^{mn}_{-\alpha}\right)^{\dagger}|Fock\rangle\,,
\end{aligned}
\tag{2.17}
$$

where $|Fock\rangle$ denotes the Fock vacuum for the fermions Λ^{mn}. It is not hard to check that (2.17) is invariant under the residual $U(1)^{N-1}$ gauge transformations.

3. Born-Oppenheimer approximation and perturbation theory

The important feature of Matrix model that makes the existence of a threshold bound state possible is flat directions of the potential. Under the gauge-fixing condition that we employed, the coordinates along the flat directions are Cartan variables D^{μ}_i. We group them together with their fermionic counterparts Λ_i. The coordinates in transverse directions are ϕ^a_{mn} along with their superpartners Λ^{mn}. Following the terminology of Born-Oppenheimer approximation we call the variables D^{μ}_i, Λ_i "slow" and the variables ϕ^a_{mn} , Λ^{mn} "fast". When slow variables are considered to be fixed the dynamics of the fast ones is governed by the oscillator Hamiltonians $H_B + H_F$ and by H_4. In the asymptotic region where the frequencies r_{mn} of the oscillators are large and assuming that H_4 can be treated as a perturbation, it is natural to expect that the fast degrees of freedom will remain in the oscillators ground state. The right assymptotic region for our purposes turns out to be $D^9_i \to \infty$. This is the same as setting $\lambda_m - \lambda_n \to \infty$ for all $m < n$. We will assume that $\lambda_m - \lambda_n\,, m < n$ are all of the order r where $r \to \infty$. Taking into account that the variables ϕ^a_{mn} in the oscillators ground state are of the order $1/\sqrt{r_{mn}}$ one can estimate that H_4 is of the order $1/\sqrt{r}$, that indeed allows one to treat it as a perturbation when $r \to \infty$.

Thus, we search for an approximate solution to the spectral problem

$$
(H - E)|\Psi\rangle = 0
$$

in the form $|\Psi\rangle = |\cdot\rangle|\Psi(D_i, \Lambda_i)\rangle$, where $|\cdot\rangle$ is the ground state of superoscillators. The general formalism for this problem was developed in [8]. We refer the reader for the detailed explanation of such generalized Born-Oppenheimer approximation to that paper. Here we just want to explain the main idea of the formalism and develop a perturbative expansion suitable for the problem at hand. If one introduces a pair of projection operators $P = |\cdot\rangle\langle\cdot|$, $Q = 1 - P$, the Schrödinger equation breaks into a system of two equations

$$
\begin{aligned}
&P(H-E)P|\Psi\rangle + P(H-E)Q|\Psi\rangle = 0\,, \\
&Q(H-E)P|\Psi\rangle + Q(H-E)Q|\Psi\rangle = 0\,.
\end{aligned}
\tag{3.1}
$$

The second equation can be formally solved as

$$
Q|\Psi\rangle = -(Q(H-E)Q)^{-1}Q(H-E)P|\Psi\rangle\,. \tag{3.2}
$$

Substituting this solution into the first equation, we get

$$
[P(H-E)P - P(H-E)Q(Q(H-E)Q)^{-1}Q(H-E)]P|\Psi\rangle\,, \tag{3.3}
$$

i.e. the "reduced" Schrodinger equation for $P|\Psi\rangle = |\cdot\rangle|\Psi(D_i, \Lambda_i)\rangle$. The first term in (3.3) is the conventional effective Hamiltonian for the Born-Oppenheimer approximation. The second one constitutes the correction term. Now note that in the problem at hand $|\cdot\rangle$ is the ground state of superoscillators that has a zero energy. If we split the total Hamiltonian as $H = H_{osc} + H'$ where H_{osc} is the superoscillators Hamiltonian and H' is all the rest, then $H_{osc}|\cdot\rangle = 0$. A direct analysis of formulae (2.4)-(2.7) shows that $QH_{osc}Q = Q(H_B + H_F)Q$ scales like r and $QH'Q$ scales like $O(1)$ (because of the part of H_0 that depends on D_i^a variables). This is the reason to treat $QH'Q$ as a perturbation. Now we can apply a perturbation theory in $QH'Q$ to the term $(Q(H-E)Q)^{-1}$ in (3.3):

$$
\begin{aligned}
(Q(H-E)Q)^{-1} &= (QH_{osc}Q + QH'Q - E)^{-1} = \\
&= \frac{1}{QH_{osc}Q - E} - \frac{1}{QH_{osc}Q - E}QH'Q\frac{1}{QH_{osc}Q - E} + \ldots
\end{aligned}
\tag{3.4}
$$

Here $Q(H-E)Q$ is understood as an operator on the Q-projected subspace of the whole Hilbert space and thus operator inverse makes sense (by the same reason we can write E instead of EQ in this equation). Substituting this perturbation expansion into reduced Schrodinger equation (3.3), we get the following expression for the effective Hamiltonian

$$
\begin{aligned}
&\langle\cdot|H'|\cdot\rangle - \langle\cdot|H'Q\frac{1}{QH_{osc}Q - E}QH'|\cdot\rangle + \\
&\quad + \langle\cdot|H'Q\frac{1}{QH_{osc}Q - E}QH'Q\frac{1}{QH_{osc}Q - E}QH'|\cdot\rangle + \ldots
\end{aligned}
\tag{3.5}
$$

The "propagator" $\frac{1}{QH_{osc}Q-E}$ scales like $1/r$ and the last expansion can be used to compute the effective Hamiltonian to any desired order in $1/r$. Although formally H' scales like $O(1)$ because of the H_0 term, the contribution of the order $1/r^2$ comes solely from the first two terms in (3.5). This happens due to the fact that $H_0|\cdot\rangle$ scales like $1/r$ (see appendix B). Hence, the correction term we need to have the effective Hamiltonian up to the order $1/r^2$ is

$$
-\langle\cdot|H_4Q\frac{1}{QH_{osc}Q - E}QH_4|\cdot\rangle\,. \tag{3.6}
$$

Another way to get the correction term is using an ansatz method similar to the one developed in [8]. Both (3.6) and the ansatz method give the same result.

4. Calculation

First we need to compute the main contribution $\langle\cdot|H|\cdot\rangle = \langle\cdot|H_0|\cdot\rangle + \langle\cdot|H_4|\cdot\rangle$ up to the second order in $1/r$. For calculations it is convenient to express H_0 in terms of λ_n^μ variables. We have a linear correspondence between the variables $D_i^\mu = A_i^n\lambda_n^\mu$. Since by definition $\sum_i D_i^\mu T^i = i\sum_i \lambda_i e_{ii}$ the condition $(D^\mu, D^\nu) = -2tr(D^\mu D^\nu) = D_i^\mu D_i^\mu$ gives $\lambda_n^\mu = \frac{1}{2}A_n^i D_i^\mu$. Therefore

$$H_0 = -\frac{1}{4}\frac{\partial^2}{\partial\lambda_n^\mu\partial\lambda_n^\mu}\,,$$

where the last operator is considered as a restriction to the subspace of functions of $\lambda_1^\mu, \ldots, \lambda_N^\mu$ which are annihilated by $\sum_n^N \frac{\partial}{\partial\lambda_n^\mu}$ (which means that they depend only on differences $\lambda_m^\mu - \lambda_n^\mu$). Both bosonic and fermionic oscillators ground state vectors depend on λ_n^μ variables. This should be taken into account when calculating averages of differential operators such as H_0. Some useful formulae for computing such averages are given in appendix B. After this preliminary work the direct computation yields

$$\langle\cdot|H_0|\cdot\rangle = -\frac{1}{2}\frac{\partial^2}{\partial D_i^\mu\partial D_i^\mu} + 9\sum_{m<n}\frac{1}{r_{mn}^2}\,. \tag{4.1}$$

In H_4 we have terms of the order $1/r$ and $1/r^{3/2}$ but it turns out that they contribute only in higher orders than $1/r^2$. The remaining terms give the answer

$$\begin{aligned}\langle\cdot|H_4|\cdot\rangle \simeq\ & \frac{7}{2}\sum_{p\neq m}\sum_{m\neq n}\frac{1}{r_{pm}r_{mn}} + \\ & +\frac{1}{2}\sum_{p\neq m\neq n\neq p}\left[\frac{2(\lambda_m-\lambda_n)(\lambda_p-\lambda_m)}{r_{mn}r_{pm}(\lambda_p-\lambda_n)^2} + \frac{r_{pm}^2+r_{mn}^2}{r_{mn}r_{pm}(\lambda_p-\lambda_n)^2}\right].\end{aligned} \tag{4.2}$$

The correction term that gives a contribution of the order $1/r^2$ is

$$\begin{aligned}&-\langle\cdot|\left(-\frac{i}{2}f_{IAB}\phi_I^a\Lambda_A\Gamma^a\Lambda_B\right)\frac{1}{H_{osc}}\left(-\frac{i}{2}f_{JCD}\phi_J^b\Lambda_C\Gamma^b\Lambda_D\right)|\cdot\rangle = \\ &= 8\sum_{p\neq m\neq n\neq p}\left[-\frac{1}{r_{mp}(r_{pm}+r_{mn}+r_{np})} + \frac{(\lambda_n-\lambda_m)(\lambda_p-\lambda_n)}{r_{mp}r_{pn}r_{nm}(r_{mp}+r_{pn}+r_{nm})}\right] - \\ &\quad -16\sum_{m<n}\frac{1}{r_{mn}^2}\,.\end{aligned} \tag{4.3}$$

Evidently the last term in (4.3) cancels with the analogous terms from (4.1) and (4.2). Those terms are similar to the ones arising in $SU(2)$ computation (see [11]). The remaining terms look, at the first glance, as if they can hardly cancel each other. To see that this indeed happens we rewrite them in terms of λ_n variables only, using the fact that

$$r_{mn} = \lambda_m - \lambda_n + \mathcal{O}\left(\frac{1}{r}\right) \,, \qquad m < n\,.$$

Then (4.2) contributes

$$\sum_{m<p<n} \left(\frac{7}{(\lambda_m - \lambda_p)(\lambda_m - \lambda_n)} + \frac{9}{(\lambda_m - \lambda_p)(\lambda_p - \lambda_n)} + \frac{7}{(\lambda_p - \lambda_n)(\lambda_m - \lambda_n)} \right).$$

The same procedure carried on the first two terms in (4.3) yields

$$-16 \sum_{m<p<n} \left(\frac{1}{(\lambda_m - \lambda_p)(\lambda_m - \lambda_n)} + \frac{1}{(\lambda_p - \lambda_n)(\lambda_m - \lambda_n)} \right).$$

One readily checks that the sum of these two contributions vanishes. Hence, the outcome of our computation is given by the formula

$$H_{eff} = -\frac{1}{2}\frac{\partial^2}{\partial D_i^\mu \partial D_i^\mu}\,.$$

In appendix B we collected some formulae which we found useful for the computation described above.

5. Discussion

First we would like to explain how the present results can be compared with those obtained in [8] for the $SU(2)$ gauge group. The authors of [8] use a gauge invariant variable R to specify the asymptotic region. Namely, R is one of the eigenvalues of matrix

$$\Phi_{ab} = \phi_a^\mu \phi_b^\mu\,.$$

Here lower a and b are $SU(2)$ indices that run from 1 two 3. Without loss of generality we can assume that index 3 corresponds to the Cartan subalgebra and the basis ϕ_a^μ is such that the matrix Φ is diagonal (any element of $SU(2)$ can be taken as the Cartan generator). Then, clearly,

$$\Phi = \begin{pmatrix} * & 0 & 0 \\ 0 & * & 0 \\ 0 & 0 & r^2 \end{pmatrix}\,,$$

where $r^2 = \phi_3^\mu \phi_3^\mu$ coincides with r_{12}^2 in our old notations. Hence, in this basis R coincides with r.

The result of [8] for the effective Hamiltonian (of radial degrees of freedom) is

$$H_{eff} = -\frac{1}{2}\frac{d^2}{dR^2} - \frac{5}{R}\frac{d}{R} - \frac{4}{R^2}\,, \tag{5.1}$$

while the measure on Hilbert space is $R^{10}dR$. Composing this operator with R from the left and with R^{-1} from the right we get the radial part of the Laplacian in 9 dimensions with the radial measure $R^8 dR$. Therefore, the radial parts of our effective Hamiltonian and the one found by Halpern and Schwartz are the same. It is not hard to check that the angular dependence is the same as well.

Once we have the effective Hamiltonian, we can find the asymptotic expression for $P|\Psi\rangle = |\cdot\rangle|\Psi(D_i, \Lambda_i)\rangle$ and then, using (3.2) and (3.4), one can get the asymptotic form of the whole state vector $|\Psi\rangle$. Asymptotic solutions to $H_{eff}|\Psi(D_i, \Lambda_i)\rangle = 0$ have a basis of the form

$$D_1^{-7-l_1} Y_{l_1}(D_1^\mu) \cdot \ldots \cdot D_{N-1}^{-7-l_{N-1}} Y_{l_{N-1}}(D_{N-1}^\mu)|\Lambda_1, \ldots, \Lambda_{N-1}\rangle\,,$$

where Y_{l_i} are SO(9) spherical harmonics. Further constraints of the supersymmetry, SO(9)-invariance and invariance with respect to the Weyl group of $SU(N)$ (i.e. the permutation group S_N) should be put on $|\Psi(D_i, \Lambda_i)\rangle$ to single out the candidates for the ground state.

Another comment we would like to make here is about the computation of Witten index. In paper [9] it was shown that Witten index for the model at hand has two contributions none of which can be made vanishing by the choice of the limiting procedure. The following expression for the correction (or surface) term was found

$$\mathcal{I}_{surf} = -\frac{1}{2}\int_{N_F(R)} tre_n(-1)^F Q W'\,.$$

The main ingredient in this formula is W' - the approximate Green's function in the limit of large separation. The authors of [9] showed that one can take a free propagator for W' in the $SU(2)$ theory. We believe that our computation justifies the same claim for $SU(N)$ case.

Acknowledgments

I would like to thank M. Halpern and C. Schwartz for helpful and stimulating discussions. I also want to express my gratitude to A. Schwarz for his constant support and interest in my work.

A. ?? Which Title ??

In this appendix we give expressions for the summands appearing in H_4 in terms of D_i^μ, $\Lambda_{i\alpha}$, ϕ^a_{mn}, Λ^{mn}_α

$$\frac{1}{4} f_{AIJ} f_{AKL} \phi^a_I \phi^b_J \phi^a_K \phi^b_L = \sum_{m\neq n\neq p\neq q\neq m} (\phi^a_{mn}\phi^a_{np}\phi^b_{pq}\phi^b_{qm} - \phi^a_{mn}\phi^b_{np}\phi^a_{pq}\phi^b_{qm})$$

$$\frac{1}{2} f_{AiJ} f_{AKL}(D^a_i \phi^b_J - D^b_i \phi^a_J)\phi^a_K \phi^b_L = \sum_{m\neq n\neq p\neq m} \frac{1}{\sqrt{2}}((\lambda^a_m - \lambda^a_n)\phi^b_{mn} - (\lambda^b_m - \lambda^b_n)\phi^a_{mn})\phi^a_{np}\phi^b_{pm}$$

$$-\frac{1}{2} f_{AiJ} f_{AkL} D^a_i D^b_k \phi^b_J \phi^a_L = \sum_{m\neq m} \frac{1}{2}(\lambda^a_n - \lambda^a_m)(\lambda^b_m - \lambda^b_n)\phi^b_{mn}\phi^a_{nm}$$

$$-\frac{i}{2}f_{IAB}\phi_I^a\Lambda_A\Gamma^a\Lambda_B = -i\sum_{m\neq n;p\neq q}[E_{mn},E_{pq}]_i\phi^a_{mn}\Lambda^{pq}\Gamma^a\Lambda_i + \frac{1}{\sqrt{2}}\phi^a_{mn}\Lambda^{np}\Gamma^a\Lambda^{pm}$$

$$\frac{1}{2}(w^tw)_{IJ}\hat{L}_I\hat{L}_J = \sum_{m\neq n}\frac{-1}{2(\lambda_n-\lambda_m)^2}\hat{L}_{nm}\hat{L}_{mn}\,,$$

where

$$\hat{L}_{nm} = \frac{i}{\sqrt{2}}\sum_p(\phi^a_{mp}\frac{\partial}{\partial\phi^a_{np}} - \phi^a_{pn}\frac{\partial}{\partial\phi^a_{pm}} + \frac{i}{\sqrt{2}}\Lambda^{mp}_\alpha\Lambda^{pn}_\alpha) -$$
$$-\sum_{p\neq q}[E_{nm},E_{pq}]_j(D^a_j\frac{\partial}{\partial\phi^a_{qp}} - \phi^a_{pq}\frac{\partial}{\partial D^a_j} + \Lambda_j\Lambda^{pq})\,.$$

B. ?? Which Title ??

When working with bosonic oscillators it is convenient to introduce the creation and annihilation operators as follows:

$$\phi^b_{mn} = \frac{a^b_{mn}+(a^b_{nm})^\dagger}{\sqrt{2r_{mn}}}\,, \qquad \frac{\partial}{\partial\phi^b_{mn}} = \frac{a^b_{nm}-(a^b_{mn})^\dagger}{\sqrt{2/r_{mn}}} \tag{B.1}$$

$$[a^b_{mn},(a^c_{pq})^\dagger] = \delta^{ab}\delta_{mp}\delta_{nq}\,, \qquad m\neq n, p\neq q\,.$$

Since both bosonic and fermionic oscillators ground states depend on eigenvalues λ^μ_m, a certain care needs to be taken when calculating the averages of differential operators over the oscillators. The following formulae are useful:

$$\frac{\partial a^b_{nm}}{\partial\lambda^\mu_k} = (\delta^{mk}-\delta^{nk})\frac{(\lambda^\mu_m-\lambda^\mu_n)}{2r^2_{mn}}(a^b_{mn})^\dagger \tag{B.2}$$

$$\frac{\partial(a^b_{nm})^\dagger}{\partial\lambda^\mu_k} = (\delta^{mk}-\delta^{nk})\frac{(\lambda^\mu_m-\lambda^\mu_n)}{2r^2_{mn}}a^b_{mn} \tag{B.3}$$

$$[\frac{\partial}{\partial\lambda^\mu_k},|\cdot\rangle] = -\sum_{m<n}(\delta_{mk}-\delta_{nk})[\frac{(\lambda^\mu_m-\lambda^\mu_n)}{2r^2_{mn}}(a^b_{nm})^\dagger(a^b_{mn})^\dagger + \tag{B.4}$$

$$+\frac{1}{2}\left(\delta^{\mu9}+\frac{(1-\delta^{\mu9})(\lambda^\mu_m-\lambda^\mu_n)}{r_{mn}+\lambda_m-\lambda_n}\right)\frac{(\lambda^a_m-\lambda^a_n)}{r^2_{mn}}(\tilde{\Lambda}^{mn})^\dagger\Gamma^a\tilde{\Lambda}^{mn} - \tag{B.5}$$

$$-\frac{1-\delta^{\mu9}}{2r_{mn}}(\tilde{\Lambda}^{mn})^\dagger\Gamma^\mu\tilde{\Lambda}^{mn}]|\cdot\rangle\,. \tag{B.6}$$

The fermionic quadratic correlator is

$$\langle 0_F|\Lambda^{mn}_\alpha\Lambda^{pq}_\beta|0_F\rangle = \frac{1}{2}\delta^{mp}\delta^{nq}\left(1-\gamma_9\frac{\lambda_m-\lambda_n}{r_{mn}}\right)_{\alpha\beta}\,, \tag{B.7}$$

for any $m\neq n$.

Note that $spin(9)$-rotated fermions $\tilde{\Lambda}^{mn}$ depend on λ^μ_m. The following correlator thus appears in computations

$$\langle 0_F|\frac{\partial}{\partial\lambda^\mu_k}(\tilde{\Lambda}^{mn})^\dagger\Gamma^a\tilde{\Lambda}^{mn}|0_F\rangle = -\frac{1}{2}Tr(\Gamma^9[R^{mn}\frac{\partial(R^{mn})^t}{\partial\lambda^\mu_k},\Gamma^a])\,. \tag{B.8}$$

References

[1] M. Claudson and M.B. Halpern, *Nucl. Phys.* **B 250** (1985) 689.

[2] R. Flume, *Ann. Phys. (NY)* **164** (1985) 189.

[3] M. Baake, P. Reinicke and V. Rittenberg, *J. Math. Phys.* **26** (1985) 1070.

[4] B. de Wit, J. Hoppe and H. Nicolai, *Nucl. Phys.* **B 305** (1988) 545.

[5] B. de Wit, M. Luscher and H. Nicolai, *Nucl. Phys.* **B 320** (1989) 135.

[6] E. Witten, *Nucl. Phys.* **B 460** (1996) 335 [hep-th/9510135].

[7] T. Banks, W. Fischler, S. Shenker and L. Susskind *M-theory as a matrix model: a conjecture*, *Phys. Rev.* **D 55** (1997) 5112 [hep-th/9610043].

[8] M. Halpern and C. Schwartz, *Asymptotic search for ground states of $SU(2)$ matrix theory*, *Int. J. Mod. Phys.* **A 13** (1998) 4367 [hep-th/9712133];

[9] S. Sethi and M. Stern, *D-brane bound states redux*, *Comm. Math. Phys.* **194** (1998) 675 [hep-th/9705046].

[10] P. Yi , *Witten index and threshold bound states of D-branes*, *Nucl. Phys.* **B 505** (1997) 307 [hep-th/9704098].

[11] J. Plefka and A. Waldron, *On the quantum mechanics of M(atrix) theory*, *Nucl. Phys.* **B 512** (1998) 460 [hep-th/9710104].

[12] M. Porrati and A. Rozenberg, *Bound states at threshold in supersymmetric quantum mechanics*, *Nucl. Phys.* **B 515** (1998) 184 [hep-th/9708119].

[13] G. Moore, N. Nekrasov and S. Shatashvili, *D-particle bound states and generalized instantons*, hep-th/9803265.

[14] J. Hoppe, *On the construction of zero energy states in supersymmetric matrix models I, II* and *III*, hep-th/9709132; hep-th/9709217; hep-th/9711033.

JHEP10(1998)018